

Theorems of operators

Theorem ①

Two eigenfunctions of a Hermitian operator with different eigenvalues are orthogonal.

Proof \Rightarrow

Let \hat{A} be the Hermitian operator. Then its eigenvalue equations are

$$\hat{A}\psi_1 = \lambda_1\psi_1 \quad \text{--- (1)}$$

$$\hat{A}\psi_2 = \lambda_2\psi_2 \quad \text{--- (2)}$$

Where ψ_1 and ψ_2 are the eigenfunctions of \hat{A} with λ_1 and λ_2 are the corresponding eigenvalues.

Taking the complex conjugate of eqn. (1) and recalling that λ_1 is real,

$$\hat{A}^*\psi_1^* = \lambda_1\psi_1^* \quad \text{--- (3)}$$

Multiplying eqn. (3) on the left by ψ_2 and integrating,

$$\int \psi_2 \hat{A} \psi_1^* d\tau = \lambda_1 \int \psi_2 \psi_1^* d\tau \quad \text{--- (4)}$$

Again, Multiplying equation (2) on the left by ψ_1^* and integrating,

$$\int \psi_1^* \hat{A} \psi_2 d\tau = \lambda_2 \int \psi_1^* \psi_2 d\tau \quad \text{--- (5)}$$

since \hat{A} is a Hermitian operator, the left hand sides of equation (4) and (5) are equal. So that

$$\lambda_1 \int \psi_2 \psi_1^* d\tau = \lambda_2 \int \psi_1^* \psi_2 d\tau$$

$$\text{or, } (\lambda_1 - \lambda_2) \int \psi_1^* \psi_2 d\tau = 0 \quad \text{--- (6)}$$

Since,

$$\lambda_1 \neq \lambda_2, \int \psi_1^* \psi_2 d\tau = 0$$

Thus,

ψ_1 and ψ_2 are orthogonal.

Theorem (2)

If two operators commute, they have the same set of eigenfunctions.

Proof \Rightarrow

Let \hat{A} be the operator whose eigenvalue equation is

$$\hat{A}\psi_i = a_i\psi_i \quad (1)$$

where ψ_i s are the set of eigenfunctions with a_i s as the corresponding eigenvalues.

Since \hat{A} and \hat{B} commute, hence,

$$\hat{A}\hat{B} = \hat{B}\hat{A} \quad (2)$$

Operating on ψ_i ,

$$\hat{A}\hat{B}\psi_i = \hat{B}\hat{A}\psi_i = \hat{B}(a_i\psi_i) = a_i(\hat{B}\psi_i) \quad (3)$$

This shows that $(\hat{B}\psi_i)$ is an eigenfunction of \hat{A} with eigenvalue a_i . This is possible only if $(\hat{B}\psi_i)$ is a multiple of ψ_i .

$$\hat{B}\psi_i = b_i\psi_i \quad (4)$$

In other words, ψ_i is also an eigenfunction of \hat{B} .